On a New Form of Quantum Mechanics (II)

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The correspondence of a new form of quantum mechanics based on a quantum version of the action principle, which was proposed earlier, with the ordinary quantum mechanics is established. New potentialities of the quantum action principle in the interpretation of quantum mechanics are considered.

PACS numbers:

I. INTRODUCTION

In the work [1] a new form of quantum mechanics based on a quantum version of the action principle was formulated for the first time. The new formulation becomes more exact in the subsequent work [2]. However, the correspondence of the new framework with the ordinary formulation of quantum mechanics remained not quite clear. The purpose of the present work is to fill in this gap. In addition, new potentialities of the quantum action principle (QAP) in the interpretation of quantum mechanics are considered.

II. QUANTUM ACTION PRINCIPLE

Let us begin with the classical action of a non-relativistic particle in a canonical form:

$$I = \int_0^T dt \left(p_k \dot{x}_k - \frac{p^2}{2m} - U(x,t) \right) \tag{1}$$

In the new form of canonical quantization procedure proposed in [1],[2], canonical variables (x_k, p_k) are represented as operators in a space of wave functionals $\Psi[x(t)]$ as follows:

$$\begin{aligned} \widehat{x}_k(t)\,\Psi &\equiv x_k(t)\,\Psi, \\ \widehat{p}_k(t)\,\Psi &\equiv \frac{\widetilde{\hbar}}{i}\frac{\delta\Psi}{\delta x_k(t)}. \end{aligned} \tag{2}$$

The constant $\widetilde{\hbar}$ is not equal to the ordinary Plank constant $\hbar$. Its physical dimensionality is $\left[\widetilde{\hbar}\right] = Joule \cdot s^2$. A relationship between two constants will be introduced here as a central point of the correspondence between two forms of quantum mecanics. The operators (2) are formally Hermitian with respect to a scalar product in a space of wave functionals:

$$(\Psi_1, \Psi_2) \equiv \int \prod_t d^3x(t)\, \overline{\Psi}_1[x(t)]\, \Psi_2[x(t)]. \tag{3}$$

The operator representation (2) permits us to define an action operator as follows:

$$\widehat{I} \equiv \int_0^T dt \left[\frac{\widetilde{\hbar}}{i} \dot{x}_k(t) \frac{\delta}{\delta x_k(t)} + \frac{\widetilde{\hbar}^2}{2m} \frac{\delta^2}{\delta x^2(t)} - U(x(t),t) \right] \tag{4}$$

The first term of the integrand (4) is non-Hermitian, however, we can overcome this problem by throwing away the corresponding imaginary parts of eigenvalues of the action operator. Two remaining terms are formally Hermitian with respect to the scalar product (3).

Let us turn to the formulation of QAP. For the action operator (4) we consider the eigenvalue problem:

$$\widehat{I}\Psi = \lambda\Psi. \tag{5}$$

The statement is that the equation (5) is an analog of Schrödinger equation. It is usefull to re-formulate the eigenvalue problem, introducing for any wave functional Ψ a functional:

$$\Lambda[x(t)] \equiv \frac{\widehat{I}\Psi[x(t)]}{\Psi[x(t)]}. \tag{6}$$

For the wave functional the exponential representation

$$\Psi[x(t)] \equiv \exp\left(\frac{i}{\widetilde{\hbar}} S[x(t)] + R[x(t)] \right) \tag{7}$$

with real functionals $S[x(t)], R[x(t)]$ will be usefull, in particular, for quasi-classical decomposition of a solution.

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Substituting (7) in (6), we obtain:

$$\Lambda[x] = \Lambda_{Re}[x] + i\widetilde{\hbar}\Lambda_{Im}[x], \quad (8)$$

$$\Lambda_{Re}[x] \equiv \int_0^T dt \left\{ \dot{x}_k \frac{\delta S}{\delta x_k} + \frac{1}{2m}\left[-\left(\frac{\delta S}{\delta x_k}\right)^2 + \widetilde{\hbar}^2 \left(\left(\frac{\delta R}{\delta x_k}\right)^2 + \frac{\delta^2 R}{\delta x_k^2} \right) \right] - U \right\}, \quad (9)$$

$$\Lambda_{Im}[x] \equiv \int_0^T dt \left[-\dot{x}_k \frac{\delta R}{\delta x_k} + \frac{1}{2m}\left(2\frac{\delta S}{\delta x_k}\frac{\delta R}{\delta x_k} + \frac{\delta^2 S}{\delta x_k^2} \right) \right]. \quad (10)$$

The necessary condition of equality of the functional (6) to an eigenvalue of the action operator is its independence on internal points $x(t), t \in (0,T)$ of a trajectory. Only dependence on the end points x_{0k}, x_{Tk}, which are fixed at this stage, is admited. Now analytical properties of the potential $U(x,t)$ will be important. We shall assume that this potential is a real-analytical function of x, i.e., it can be represented by a series:

$$U(x,t) = U_0(t) + U_{1k}(t)\, x_k + \frac{1}{2} U_{2kl}(t)\, x_k x_l + ... \quad (11)$$

Then the functionals $S[x(t)], R[x(t)]$ can be represented as functional series:

$$S[x(t)] = \int_0^T dt s(x(t),t), \quad (12)$$

$$s(x(t),t) \equiv s_{1k}(t)\, x_k(t) + \frac{1}{2} s_{2kl}(t)\, x_k(t)\, x_l(t) + ...,$$

$$R[x(t)] = \int_0^T dt r(x(t),t), \quad (13)$$

$$r(x(t),t) \equiv r_{1k}(t)\, x_k(t) + \frac{1}{2} r_{2kl}(t)\, x_k(t)\, x_l(t) + ...$$

Let us outline that the correspondence with ordinary quantum mechanics will take place on a narrow class of wave functionals with local in time functionals (12), (13). Substituting decompositions (12), (13) in (9), (10), and integrating by parts of the first terms under integrals, we obtain:

$$\Lambda_{Re}[x] = s(x(t),t)|_0^T - \int_0^T dt \left\{ \dot{s} + \frac{1}{2m}\left(\frac{\partial s}{\partial x_k}\right)^2 + U - \frac{\widetilde{\hbar}^2}{2m}\left[\left(\frac{\partial r}{\partial x_k}\right)^2 + \frac{\partial^2 r}{\partial x_k^2} \right] \right\}, \quad (14)$$

$$\Lambda_{Im}[x] = -r(x(t),t)|_0^T + \int_0^T dt \left[\dot{r} + \frac{1}{2m}\left(2\frac{\partial s}{\partial x_k}\frac{\partial r}{\partial x_k} + \frac{\partial^2 s}{\partial x_k^2} \right) \right]. \quad (15)$$

We must put equal to zero all coefficients in front of non-zero degrees of $x_k(t)$ in the integrands (14), (15). This necessary condition leads to an infinite system of differential equations for coefficients of the series (12) and (13):

$$\begin{aligned}
&\dot{s}_{1k} + \frac{1}{m} s_{1l} s_{2lk} - \frac{\widetilde{\hbar}^2}{2m}\left(2 r_{1l} r_{2lk} + r_{3llk}\right) + U_{1k} = 0,\\
&\dot{s}_{2kl} + \frac{1}{m}\left(s_{2mk} s_{2ml} + 2 s_{1m} s_{3mkl}\right) - \frac{\widetilde{\hbar}^2}{m}\left(r_{2mk} r_{2ml} + 2 r_{1m} r_{3mkl}\right) + U_{2kl} = 0,\\
&...,\\
&\dot{r}_{1k} + \frac{1}{m}\left(s_{1l} r_{2lk} + r_{1l} s_{2lk}\right) + \frac{1}{2m} s_{3llk} = 0,\\
&\dot{r}_{2kl} + \frac{2}{m}\left(s_{1m} r_{3mkl} + s_{2mk} r_{2ml} + s_{3mkl} r_{1m}\right) + \frac{1}{m} s_{4mmkl} = 0,\\
&...,
\end{aligned} \quad (16)$$

It is the system that, with two additions, is equivalent to the Schrödinger equation. These additions arise from the coefficients in front of zero degrees of x_k in the integrands (14) and (15). The first one appears in the imaginary part (15) of the functional (6), and it has to be equal zero:

$$\int_0^T dt \left(2 s_{1k} r_{1k} + s_{2kk}\right) = 0, \quad (17)$$

if the action operator is Hermitian. The remaining part of (15) we omit in accordance with the notion made after the (4). The second addition appears from the real part (14) of the functional (6):

$$f(t) \equiv \frac{1}{2m} s_{1k}^2 - \frac{\widetilde{\hbar}^2}{2m}\left(r_{1k}^2 + r_{2kk}\right) + U_0. \quad (18)$$

This function is not equal zero. On the one hand, it will be added to the Hamiltonian operator of the Schrödinger theory. This addition does not change the physical content of the theory. On the other hand, it will be a part of an action eigenvalue:

$$\lambda = \Lambda_{Re} = s|_0^T + \int\limits_0^T dt f(t), \tag{19}$$

III. CORRESPONDENCE OF QAP WITH ORDINARY QUANTUM MECHANICS

To establish this correspondence, let us consider a multiplicative representation of a wave functional [1]. For this purpose, we divide the interval of time $[0,T]$ into N small parts of an equal length $\varepsilon = T/N$, and approximate a trajectory $x_k(t)$ by a piecewise linear function with vertices $x_k(t_n), t_n = n\varepsilon$; $x_k(0) = x_{0k}$, $x_k(T) = x_{Tk}$. Then the functionals (12) and (13) can be approximated by corresponding integral sums:

$$S[x] = \sum_{n=1}^{N} \varepsilon s(x_n, t_n), R[x] = \sum_{n=1}^{N} \varepsilon r(x_n, t_n). \tag{20}$$

The central point of the correspondence between two forms of quantum mechanics is the equality [1]:

$$\widetilde{\hbar} = \varepsilon\hbar. \tag{21}$$

Taking into account Eqs. (20) and (21), the exponential representation of a wave functional (7) can be transformed to the product of wave functions taken at discrete moments of time:

$$\begin{aligned} \Psi[x] &= \prod_{n=1}^{N} \psi(x_n, t_n), \psi(x_n, t_n) \equiv \exp\chi(x_n, t_n). \\ \chi(x_n t_n) &\equiv \frac{i}{\hbar} s(x_n, t_n) + \varepsilon r(x_n, t_n) \end{aligned} \tag{22}$$

Here and further the product εr will be considered as a single symbol. In this approximation a wave functional Ψ is a function of many variables - coordinates of vertices $x_k(n)$ of a broken line, and its variational derivative has to be replaced by the partial derivative as follows [1]:

$$\frac{\delta\Psi}{\delta x_k(t_n)} \equiv \frac{1}{\varepsilon}\frac{\partial\Psi}{\partial x_k(t_n)} = \frac{1}{\varepsilon}\frac{\partial\psi(x_n, t_n)}{\partial x_k(t_n)}. \tag{23}$$

Then the action operator may be approximated by the differential operator:

$$\begin{aligned} \widehat{I}\Psi &= \sum_{n=1}^{N} \varepsilon \left[\frac{\hbar}{i}\frac{x_k(t_n) - x_k(t_{n-1})}{\varepsilon}\frac{\partial\Psi}{\partial x_k(t_n)} \right. \\ &\left. + \frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x_k^2(t_n)} - U(x_n, t_n)\Psi \right]. \end{aligned} \tag{24}$$

The first of three parts in the right hand side of Eq. (24) can be transformed as follows:

$$\begin{aligned} &\sum_{n=1}^{N} \varepsilon\frac{\hbar}{i}\frac{x_k(t_n) - x_k(t_{n-1})}{\varepsilon}\frac{\partial\Psi}{\partial x_k(t_n)} \\ &= \sum_{n=1}^{N} \varepsilon\frac{\hbar}{i}\frac{x_k(t_n) - x_k(t_{n-1})}{\varepsilon}\frac{\partial\chi(x_n, t_n)}{\partial x_k(t_n)}\Psi \\ &\cong \sum_{n=1}^{N} \varepsilon\frac{\hbar}{i}\left[\frac{\chi(x_n, t_n) - \chi(x_{n-1}, t_{n-1})}{\varepsilon} - \frac{\partial\chi(x_n, t_n)}{\partial t_n}\right]\Psi \\ &= \left[\frac{\hbar}{i}\chi(x_n, t_n)\Big|_{n=0}^{n=N} - \sum_{n=1}^{N} \varepsilon\frac{\hbar}{i}\frac{\partial\chi(x_n, t_n)}{\partial t_n}\right]\Psi \end{aligned} \tag{25}$$

The approximate equality becomes exact in the limit $N \to \infty$, which is supposed. The functional (7) in this approximation takes a form:

$$\Lambda[x] = \lambda + \sum_{n=1}^{N} \varepsilon\frac{\check{S}ch\psi(x_n, t_n)}{\psi(x_n, t_n)}, \tag{26}$$

where λ is an eigenvalue of the action operator (24) approximated as follows:

$$\lambda \equiv s(x_T, T) - s(x_0, 0) + \sum_{n=1}^{N} \varepsilon f(t_n), \tag{27}$$

f is defined by (18), and

$$\check{S}ch\psi \equiv i\hbar\dot{\psi} + \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} - U\psi - f\psi = 0 \tag{28}$$

is a part of the functional, which depends on non-zero degrees of $x_k(t)$. Here we have omitted all imaginary contributions in accordance with notions made before. Therefore, QAP reduces in this approximation to the Schrödinger equation for a wave function with addition to the Hamiltonian a function of time defined by (18). Let us remember that this approximation is exact in the limit $N \to \infty$.

The multiplicative representation of a wave functional (22) and its connection with a solution of Schrödinger equation (28) gives us a simple instruction for the probabilistic interpretation of our approach: a wave functional $\Psi[x]$ in the discrete approximation (22) is a complex amplitude of probability of a particle movement along a broken line between the end points x_{0k}, x_{Tk}. More precise, if $[x_n - \delta_n, x_n + \delta_n]$ is an interval in a neighborhood of the vertice x_n, then the probability of particle movement along a broken line from that neighborhood is given by the expression:

$$P_\delta[x] = \int\limits_{x_0-\delta_0}^{x_0+\delta_0} d^3x_0 \int\limits_{x_1-\delta_1}^{x_1+\delta_1} d^3x_1 ... \int\limits_{x_T-\delta_T}^{x_T+\delta_T} d^3x_T \left|\Psi[x_n]\right|^2. \tag{29}$$

However, this possibility of the probabilistic interpretation is not quite correct. Any attempt to localize the electron in intermediate points perturbs its posterior movement. Physically correct is the probability of the electron localization in a neighborhood of the end points:

$$\begin{aligned} & P\left[|x-x_0| \leq \delta_0, |x-x_T| \leq \delta_T\right] \\ = & \int\limits_{x_0-\delta_0}^{x_0+\delta_0} d^3x_0 \int\limits_{R^3} d^3x_1 ... \int\limits_{R^3} d^3x_{N-1} \int\limits_{x_T-\delta_T}^{x_T+\delta_T} d^3x_T \left|\Psi\left[x_n\right]\right|^2 \\ = & \int\limits_{x_0-\delta_0}^{x_0+\delta_0} d^3x_0 \left|\psi\left(x_0,0\right)\right|^2 \int\limits_{x_T-\delta_T}^{x_T+\delta_T} d^3x_T \left|\psi\left(x_T,T\right)\right|^2 \end{aligned} \quad (30)$$

In the limit $N \to \infty$ we arrive at a functional integral over a space of trajectories with partly fixed end points. In the next section other possibilities of interpretation of the new framework are considered.

IV. QAP IN A STRONG FORM

The Schrödinger equation (28), in fact, is a nonlinear equation, in so far as the function $f\left(t\right)$, according to (18), depends on ψ. Though the physical content of the Schrödinger theory does not changes with addition to the Hamiltonian an arbitrary function of time, this term is important for the correspondence of QAP with ordinary quantum mechanics. At the same time, this function enters in the eigenvalue λ of the action operator. Let us focus our attention on the physical meaning of the eigenvalue λ. Being a function of a solution of Schrödinger equation, λ may be considered as a generation function for "observables". Let us introduce for this purpose a probe field $A_\mu\left(x,t\right)$. For instance, it may be an electromagnetic field. Then λ becomes a functional of $A_\mu\left(x,t\right)$. It is this functional that generates currents and their correlators in electrodynamics. Precisely,

$$\left\langle j^\mu\left(x,t\right)\right\rangle \equiv \frac{\delta\lambda}{\delta A_\mu\left(x,t\right)} \quad (31)$$

is a mean value of the electron current density for a given solution of the Schrödinger equation. Correlators of currents are equal to higher order variational derivatives. These "observables" give us information about the electron movement in a "soft" form. However, this form of observation also works up to the moment when the probe field will perturb the electron movement.

Finally, we propose more questionable interpretation of eigenvalues of the action operator, based on a strong form of QAP, which was formulated in [1],[2]. In our framework λ is open dependent on phases $s\left(x_0,0\right)$ and $s\left(x_T,T\right)$ of initial and final wave functions taken at the end points x_{0k}, x_{Tk}. These phases are represented by series:

$$\begin{aligned} s\left(x_0,0\right) &= s_{1k}\left(0\right)x_{0k} + \frac{1}{2}s_{2kl}\left(0\right)x_{0k}x_{0l} + ..., \\ s\left(x_T,T\right) &= s_{1k}\left(T\right)x_{Tk} + \frac{1}{2}s_{2kl}\left(T\right)x_{Tk}x_{Tl} + ... \end{aligned} \quad (32)$$

In the strong form of QAP an additional condition of stationarity of an eigenvalue λ with respect to small variations of the coefficients $s_{1k}\left(0\right), s_{2kl}\left(0\right), ..., r_{1k}\left(0\right), r_{2kl}\left(0\right), ...$ was accepted. These coefficients play a role of initial data for the system of differential equations (16), or, equivalently, for the Schrödinger equation (28). The stationary eigenvalue λ_0 becomes a function only of time and coordinates of the end points: $\lambda_0 = \lambda_0\left(x_0, x_T, T\right)$. What is the physical meaning of this function? In the work [2] the classical limit $\hbar \to 0$ of this function in the case of a harmonic oscillator was obtained: it coincides with the classical action calculated along a stationary trajectory of a particle between the end points x_{0k} and x_{Tk}. We shall suppose that this limit takes place in general case. Therefore, the stationary eigenvalue λ_0 may be considered as a quantum analog of the classical action $I\left(x_0, x_T, T\right)$. This statement has no practical meaning if we don't make the next step. In classical mechanics the action $I\left(x_0, x_T, T\right)$ is a generating function of the canonical transformation from the initial canonical variables $\left(x_0, p_0\right)$ to the final ones $\left(x_T, p_T\right)$, which is defined by the equations (see, for example, [3]):

$$\frac{\partial I\left(x_0,x_T,T\right)}{\partial x_{0k}} = -p_{0k}, \frac{\partial I\left(x_0,x_T,T\right)}{\partial x_{Tk}} = p_{Tk}. \quad (33)$$

These equations gives us coordinates of the end point as a function of time and the initial data: $x_{Tk} = x_{Tk}\left(x_{0k}, p_{0k}, T\right)$. It is a solution of classical equations of motion. Our proposal is to consider a quantum analog of Eq. (33):

$$\frac{\partial \lambda_0\left(x_0,x_T,T\right)}{\partial x_{0k}} = -p_{0k}, \frac{\partial \lambda_0\left(x_0,x_T,T\right)}{\partial x_{Tk}} = p_{Tk}. \quad (34)$$

If the equations (34) have a unique solution for coordinates of the end point $x_{Tk} = x_{Tk}\left(x_{0k}, p_{0k}, T\right)$, one can interpret it as a causal prediction for an electron movement in the framework of a theory of hidden parameters analogous to the D.Bohm theory [4]. Here the initial data $\left(x_{0k}, p_{0k}\right)$ form a set of hidden parameters with a probabilistic measure defined by an initial quantum state of the system.

V. CONCLUSIONS

In conclusion, we showed that the new formulation of quantum mechanics based on a quantum version of the action principle, taken on a narrow class of exponential wave functionals, is equivalent to the ordinary Schrödinger formulation of quantum mechanics. This

new approach gives new potentialities for interpretation of quantum mechanics. In addition to the usual probabilistic interpretation in terms of a wave function (or a wave functional) one obtains a quantum analog of the classical action, which may be used as a basis of a theory of hidden parameters.

We thank V. A. Franke and A. V. Goltsev for useful discussions.

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[1] N. N. Gorobey, and A. S. Lukyanenko, arXiv: 0807.3508 (2008).
[2] N. N. Gorobey, and A. S. Lukyanenko, arXiv: 0810.2255 (2008).
[3] J. L. Singe, *Classical Dynamics*, (edited by S. FL. GGE/MARBURG, Berlin-Göttingen-Heidelberg,1960)
[4] D. Bohm, and B. J. Hiley, *The undivided universe*, (edited by Routledge, London and New York, 1993).